AIAA JOURNAL
Vol. 33, No. 7, July 1995

Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Geometric Properties of Arbitrary Polyhedra in Terms of Face Geometry

Christopher W. S. Bruner*
Naval Air Warfare Center,
Patuxent River, Maryland 20670-5304

Introduction

THE calculation of cell volumes in a finite volume computational fluid dynamics (CFD) code is necessary in order to update the solution variables; it is often desirable to compute the centroids as well. Whereas relations exist for tetrahedra, no general expressions for the volume or centroid of an N-faced solid seem to exist in the literature. Hirsch[1] suggests using the present method to find the volume of a tetrahedron and pyramid but suggests using the average of two different tetrahedral decompositions for the volume of a hexahedron; the centroid calculation is not addressed. The distinguishing feature of the present method is that no interpolation from the corner points is necessary to evaluate the volume; hence, a CFD code using this method to compute cell properties need not limit itself to a restricted range of cell topologies.

Formulation

Expression for the Volume

Recall the divergence theorem[2,3] from vector calculus

$$\int_{\Omega} \nabla \cdot \boldsymbol{x}\, \mathrm{d}V = \oint_{S} \boldsymbol{x} \cdot \boldsymbol{n}\, \mathrm{d}S \tag{1}$$

where $\boldsymbol{x}$ is continuous. If we choose the position vector $\boldsymbol{r}$ as the vector $\boldsymbol{x}$ in Eq. (1), the left-hand side becomes simply

$$\int_{\Omega} \nabla \cdot \boldsymbol{r}\, \mathrm{d}V = \int_{\Omega} \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) \mathrm{d}V = 3\Omega \tag{2}$$

Accordingly, if the right-hand side of Eq. (1) can be evaluated exactly, then the volume of any solid may be evaluated exactly. For solids with triangular faces, the right-hand side becomes

$$\oint_{S} \boldsymbol{r} \cdot \boldsymbol{n}\, \mathrm{d}S = \sum_{i=1}^{N\ \mathrm{faces}} \boldsymbol{r}_i \cdot \boldsymbol{n}_i S_i \tag{3}$$

where $\boldsymbol{r}_i$ is the position vector of the centroid of the face, and $\boldsymbol{n}_i$ and S_i are the outward-directed unit normal vector and area of face i, respectively.

If the solid whose volume is to be computed has quadrilateral faces, then we can define the area, normal, and centroid as the area-weighted average of the two different triangular decompositions of the face. The volume computed using this definition in Eq. (3) is equivalent to the average of the volumes computed by decomposing each quadrilateral face into both sets of two triangles. The advantage is that, once the face data has been computed, the face topology is no longer needed; hence, volumes may be computed solely on the basis of face area, center, and normal, i.e., the volume of a cell is independent of the number and topology of its surrounding faces:

$$\Omega = \frac{1}{3} \sum_{i=1}^{N\ \mathrm{faces}} \boldsymbol{r}_i \cdot \boldsymbol{n}_i S_i \tag{4}$$

Expression for the Centroid

The centroid of any solid is given by

$$\boldsymbol{r}_c = \frac{1}{\Omega} \int_{\Omega} \boldsymbol{r}\, \mathrm{d}V \tag{5}$$

The problem is how to evaluate the integral in Eq. (5) for an arbitrary solid. From the gradient theorem[2]

$$\int_{\Omega} \nabla \varphi\, \mathrm{d}V = \oint_{S} \varphi \boldsymbol{n}\, \mathrm{d}S \tag{6}$$

If we can express $\boldsymbol{r}$ as the gradient of some scalar function, then we can express Eq. (5) in terms of an integral over the surface. The expression for φ whose gradient is $\boldsymbol{r}$ is

$$\varphi = \tfrac{1}{2}(x^2 + y^2 + z^2) = \tfrac{1}{2}\boldsymbol{r} \cdot \boldsymbol{r} \tag{7}$$

So the position vector of the centroid becomes

$$\boldsymbol{r}_c = \frac{1}{2\Omega} \oint_{S} (x^2 + y^2 + z^2)\boldsymbol{n}\, \mathrm{d}S \tag{8}$$

For triangles, Eq. (8) can be evaluated exactly using three-point Gaussian quadrature; quadrilaterals require four points.

Conclusions

General expressions for the volume and centroid of an arbitrary polyhedral cell have been derived. It is shown that these quantities can be computed in a way that is independent of the topology of the cell and its surrounding faces. If the face centers, normals, and areas are computed appropriately, then the resulting expression for the volume is exact and requires no other data; however, numerical integration over the face must be used to compute the centroid. In particular, the volume and centroid for an arbitrary element composed of triangular cell faces is given exactly.

Acknowledgment

This work was supported by the Office of Naval Research through the In-house Laboratory Independent Research (ILIR) Program.

References

[1] Hirsh, C., *Numerical Computation of Internal and External Flows*, Vol. 1, Wiley, New York, 1988, pp. 256–260.

[2] Karamcheti, K., *Principles of Ideal-Fluid Aerodynamics*, Krieger, Malabar, FL, 1980, pp. 129–131.

[3] Aris, R., *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*, Dover, Mineola, NY, 1989, p. 58.

Received Sept. 29, 1994; revision received Dec. 8, 1994, accepted for publication Dec. 22, 1994.

*Aerospace Engineer, Aerodynamics and Performance Branch, Code 4.3.2.1, Mail Stop 2, Bldg. 1403. Student Member AIAA.